Lecture #16

CS 170

Spring 2021

Zero Sum Games

Ex: Rock-Paper-Scissors (R-P-S)

2 Players (call them Row and Col)

Both pick one of R, P, or S; who wins?

Col

Row:	R	P	S
R			
P			
S			

each enty of "utility matrix" says how much Row wins, = how much Col loses (reason for name "Zero Sum")

Row wants to value of entry

Col wants to value of entry

What is best strategy for Row? for Col?

What does "play the game" mean? (1/3)

Row:

	Col R	Col P	Col S
R	0	-1	1
P	1	0	-1
S	-1	1	0

$\begin{bmatrix}\text{Row}\\ \text{Col}\end{bmatrix}$ wants to $\begin{bmatrix}\text{maximize}\\ \text{minimize}\end{bmatrix}$ entry

1) Row picks R, P or S, tells Col, then Col picks

2) Col picks R, P or S, tells Row, then Row picks

3) Row and Col pick, then announce at same time

What does "play the game" mean? (2/3)

$$
\text{Row: }\quad
\begin{array}{c|ccc}
 & \multicolumn{3}{c}{\text{Col}} \\
 & R & P & S \\
\hline
R & 0 & -1 & 1 \\
P & 1 & 0 & -1 \\
S & -1 & 1 & 0
\end{array}
\qquad
\begin{bmatrix}\text{Row}\\ \text{Col}\end{bmatrix} \text{ wants to } \begin{bmatrix}\text{maximize}\\ \text{minimize}\end{bmatrix} \text{ entry}
$$

3) Row and Col pick, then announce at same time

3a) Row (almost) always picks same row

3b) Col (almost) always picks same column

What does "play the game" mean? (3/3)

Row:

	Col: R	P	S
R	0	-1	1
P	1	0	-1
S	-1	1	0

$\begin{bmatrix}\text{Row}\\ \text{Col}\end{bmatrix}$ wants to $\begin{bmatrix}\text{maximize}\\ \text{minimize}\end{bmatrix}$ entry

3) Row and Col pick, then announce at same time

Row picks row i with probability x_i

Col picks column j with probability y_j

Let $U = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}$ = "utility"

Probability of picking $U(i,j)$ =

Expected utility = $\mathbb{E}U$ =

Choosing a "strategy" for game

- Row's strategy $= x = (x_1, x_2, x_3)$ $\quad U = \begin{array}{cc} \begin{array}{ccc} R & P & S \end{array} & \\ \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} & \begin{array}{c} R \\ P \\ S \end{array} \end{array}$

  Col's strategy $= y = (y_1, y_2, y_3)$
- Ex: $x = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, $y = (0, 0, 1)$

- Ex: $x = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, $y = (y_1, y_2, y_3)$

- Similarly, if $x = (x_1, x_2, x_3)$, $y = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ then
- Both Row and Col have strategies that guarantee the other player wins in expectation
- Value of game =

Let's try a different game

		L	R
Row	T	5	-3
	B	-1	1
		Col	

- Is $x = (\frac{1}{2}, \frac{1}{2})$ still a good strategy for Row?

  if Col picks $\begin{bmatrix} L \\ R \end{bmatrix}$ then $\mathbb{E}U = \begin{bmatrix} \quad \end{bmatrix}$

  $\Rightarrow$ Col should choose

- Is $y = (\frac{1}{2}, \frac{1}{2})$ still a good strategy for Col?

  if Row picks $\begin{bmatrix} T \\ B \end{bmatrix}$ then $\mathbb{E}U = \begin{bmatrix} \quad \end{bmatrix}$

  $\Rightarrow$ Row should choose

- Not like RPS, expect a better strategy

Let's try a different game

$$\begin{array}{cc} & \begin{array}{cc} L & R \end{array} \\ \text{Row} \begin{array}{c} T \\ B \end{array} & \begin{bmatrix} 5 & -3 \\ -1 & 1 \end{bmatrix} \\ & \text{Col} \end{array}$$

Are there better strategies for Row and Col than $\left(\frac{1}{2}, \frac{1}{2}\right)$?

- Suppose Row plays $x = \left(\frac{1}{5}, \frac{4}{5}\right)$

  if Col picks $\begin{bmatrix} L \\ R \end{bmatrix}$ then $\mathbb{E}U = \begin{bmatrix} \quad \\ \quad \end{bmatrix}$

  Row wins

- Suppose Col plays $y = \left(\frac{2}{5}, \frac{3}{5}\right)$

  if Row picks $\begin{bmatrix} T \\ B \end{bmatrix}$ then $\mathbb{E}U = \begin{bmatrix} \quad \\ \quad \end{bmatrix}$

  Col loses

- Value of game =

Solving a Zero Sum Game as a LP (1/2)

		L	R
Row	T	5	-3
	B	-1	1
		Col	

Rephrase finding the best strategies for Row and Col as dual LPs

- Row's goal: choose $x = (x_1, x_2)$ to maximize payoff from Col's best response: pick x to
- Convert to LP:
  constraints:

  compute

  LP:

Solving a Zero Sum Game as a LP (2/2)

		L	R
Row	T	5	-3
	B	-1	1
		Col	

Rephrase finding the best strategies for Row and Col as dual LPs

- Col's goal: choose $y = (y_1, y_2)$ to minimize payoff from Row's best response:
  pick y to
- Convert to LP:
  constraints:

  compute

  LP:

These Two LPs are Dual

- Row's LP: maximize

  $z = 1 \cdot z + 0 \cdot x_1 + 0 \cdot x_2$ s.t.

  $x_1 \geq 0,\ x_2 \geq 0,\ x_1 + x_2 = 1,$

  $z \leq 5x_1 - x_2,\ z \leq -3x_1 + x_2$

- Col's LP: minimize

  $w = 1 \cdot w + 0 \cdot y_1 + 0 \cdot y_2$ s.t.

  $y_1 \geq 0,\ y_2 \geq 0,\ y_1 + y_2 = 1$

  $w \geq 5y_1 - 3y_2,\ w \geq -y_1 + y_2$

- Write $z = z_1 - z_2,\ z_1 \geq 0, z_2 \geq 0$

  $$\max z = [1, -1, 0, 0] \begin{bmatrix} z_1 \\ z_2 \\ x_1 \\ x_2 \end{bmatrix} =$$

  s.t.

- Write $w = w_1 - w_2,\ w_1 \geq 0, w_2 \geq 0$

  $$\min w = [0, 0, 1, -1] \begin{bmatrix} y_1 \\ y_2 \\ w_1 \\ w_2 \end{bmatrix} =$$

  s.t.

$z_1 - z_2 - 5x_1 + x_2 \leq 0$

$z_1 - z_2 + 3x_1 - x_2 \leq 0$

$$\begin{bmatrix} \ & \ & \ & \ \\ \ & \ & \ & \ \end{bmatrix} \cdot \begin{bmatrix} z_1 \\ z_2 \\ x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} \ \\ \ \end{bmatrix} \qquad \begin{bmatrix} \ & \ & \ & \ \\ \ & \ & \ & \ \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \\ w_1 \\ w_2 \end{bmatrix} \geq \begin{bmatrix} \ \\ \ \end{bmatrix}$$

max s.t. min s.t.

General Approach to Zero Sum Games

- Game specified by $m \times n$ matrix U

  Row strategy $= x = (x_1, x_2, \dots x_m)^T$, $x_i \geq 0$, $\sum_i x_i = 1$

  Col strategy $= y = (y_1, y_2, \dots, y_n)^T$, $y_j \geq 0$, $\sum_j y_j = 1$

- Expected Utility $= \mathbb{E}U = \sum_{i,j} U(i,j) \cdot x_i \cdot y_j = x^T U y$

Row's Goal: pick x to maximize $\min(x^T U)$

Col's Goal: pick y to minimize $\max(Uy)$

Game Theory - A little history

- Von Neumann — before duality of LPs
- Many variations:

- Many Applications

- Many Prizes